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A Simple, Accurate, Geometrical Approximation to the Keplerian Motion

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A new method which shows, simply and accurately, the planetary position as a geometrical point in the orbit as a function of time is presented. By comparison with the historical hypotheses of planetary motion, it is thus visually recognizable how such hypotheses, especially those propagated in the 17th century, approached the Keplerian motion geometrically by means of observations. The reason why such a mathematically accurate hypothesis as that presented here was not developed previously, say in the 17th century, was due mainly to the inaccurate values for the solar parallax involved in the observations of that time. © 1989 Academic Press, Inc.

Dargelegt wird eine neue Methode, die die Planetenposition, einfach und genau, als einen geometrischen Punkt in der Bahn als Funktion der Zeit darstellt. Durch den Vergleich mit den historischen Hypothesen zur Planetenbewegung ist es daher visuell erkennbar, wie sich diese, insbesondere die aus dem 17. Jh., anhand der Beobachtungen geometrisch an die Kepler-Bewegung annäherten. Der Grund dafür, weshalb in der Geschichte, z. B. im 17. Jh., eine solche mathematisch genaue Hypothese, wie die hier gezeigte, nicht entstand, lag vorwiegend in den ungenauen Werten für die Sonnenparallaxe, die mit den Beobachtungen jener Zeit verquickt waren. © 1989 Academic Press, Inc.

On présente ici une nouvelle méthode qui montre, de façon simple et précise, la position planétaire en tant qu'un point géométrique dans l'orbite en fonction du temps. Par comparaison avec les hypothèses historiques au mouvement planétaire il est ainsi visuellement reconnaissable qu'elles, en particulier celles du XVII^{e} siècle, s'approchèrent du Kepler-mouvement géométriquement au moyen des observations. La raison pour laquelle une telle hypothèse mathématiquement précise ne fut pas dévelopée, par exemple, au XVII^{e} siècle, réside principalement aux valeurs imprécises pour la parallaxe solaire qui étaient inclues dans les observations de ce temps-là. © 1989 Academic Press, Inc.

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INTRODUCTION

The true anomaly of a planet, V_K, as given by Gauss's and Kepler's equations

$$\tan(V_K/2) = \sqrt{(1 + e)/(1 - e)}\,\tan(E/2), \qquad \text{where } \alpha = E - e \sin E,$$

cannot be expressed in a finite number of terms as a function of the mean anomaly α. Nor is it possible to display the planetary position in the orbit geometrically, since the second equation shown above is transcendental in the eccentric anomaly, E.

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By the method presented below, the true planetary position can be determined very simply at any given instant to within, e.g., in the case of Mars, one-half of a second of arc [1], and, if a further geometrical step is taken, to within one-tenth thereof, 0.05″. This is thus presented as one of the simplest and most accurate approximations proposed thus far to represent the Keplerian motion as a geometrical point in the orbit as a function of time [2].

(1) Geometrical Method 1

Applying the given mean anomaly α to the empty focus F′ [3], we draw a line, DG, through two points: (1) the intersection of the vertical line IC to the apsidal line AP with the auxiliary circle, D—the auxiliary point introduced by Boulliau [1657]—and (2) a point on the apsidal line, G, at distance g from the center O (Fig. 1). The angle DGF $= \delta_g$ is then given in terms of the mean anomaly α as

$$\tan \delta_g = \rho \sin \alpha / [\sqrt{1 - e^2}\,(\rho \cos \alpha - e - g)]$$
$$= \sqrt{1 - e^2} \sin(E - e \sin E)/[(1 + eg) \cos(E - e \sin E) - (e + g)], \quad (1)$$

where α is expressed according to Kepler's equation as a function of E. Likewise we may represent the true planetary position δ_K as seen from the same point G by [4]

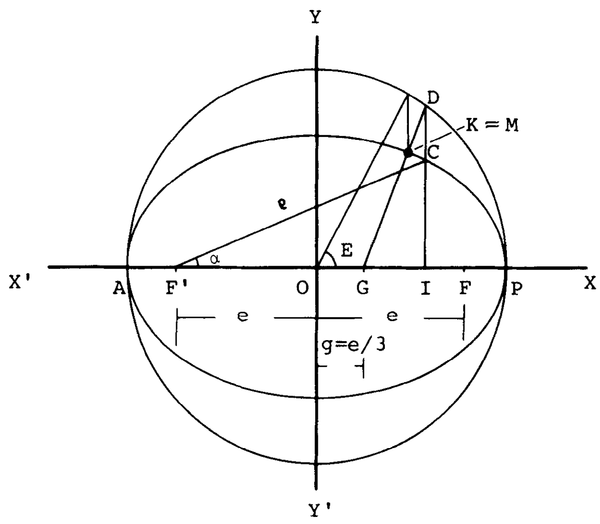

FIG. 1. PCKMA, elliptic planetary orbit; PDA, auxiliary circle; F, F′, foci of the orbit; α, the given mean anomaly applied to F′; ρ, F′C; $K(\cos E, \sqrt{1 - e^2} \sin E)$, true planetary position as given by Kepler's equation; $D(\rho \cos \alpha - e, \rho \sin \alpha/\sqrt{1 - e^2})$, auxiliary point introduced by Boulliau (1657). The geometrical point M, the intersection of the line DG (OG $= e/3$) with the orbit, represents the true planetary position accurately to the e^3-order. The error, seen from any point between F and F′, is bounded by the e^4-order, in the case of Mars by ca. 0.6″.

$$\tan \delta_K = \sqrt{1 - e^2} \sin E/(\cos E - g). \tag{2}$$

From (1) and (2) we obtain the deviation as a function of e and E:

$$\tan(\delta_g - \delta_K) = \frac{\sqrt{1 - e^2} \sin^3 E \left(\frac{e^3}{6} - \frac{e^2 g}{2} + \frac{e^3 g}{3} \cos E + \cdots\right)}{(\cos E - g)(\ldots) + \cdots},$$
$$= f(e, E). \tag{3}$$

Letting (3) = 0 we obtain

$$g = \frac{e}{3} + \frac{2}{9} e^2 \cos E + f(e^3) + \cdots. \tag{4}$$

Putting the first constant term, $g = e/3$, into (3) we may deduce the error to the e^4-approximation as

$$\delta_{g=e/3} - \delta_K \simeq \frac{e^4}{9} \sin^3 E \cos E \simeq \frac{e^4}{9} \sin^3\alpha \cos \alpha;$$

hence,

$$(\delta_g - \delta_K)_{\text{max,min}} \simeq \pm\frac{\sqrt{3}}{48} e^4 \simeq \pm 0.56'' \text{ (Mars)},$$
$$\text{at } E \simeq \alpha \simeq \pm\pi/3,\ \pi \pm \pi/3. \tag{5}$$

Since in the range between two foci, F and F′, all corresponding angular deviations consist of e-terms, the above value is also valid for the deviations as seen from two foci as

$$\delta_g - \delta_K \simeq \alpha_g - \alpha_K \simeq v_g - v_K. \tag{6}$$

Consequently the true planetary position K as given by Kepler's equation is accurately replaceable by the point M of the intersection of the line DG (OG = $e/3$ on the line OF) with the elliptic orbit, the point which can readily be found geometrically for a given mean anomaly applied to the empty focus F′. The resulting angular error, seen from any point between F and F′, is bounded by the quantity of the e^4-order, in the case of Mars by one-half of a second.

(2) Geometrical Method 2

To the same problem above we apply another method. We draw a line through D and K (Fig. 1) which intersects the X-axis at G′, whose distance g' from the center O is given by

$$g' = [e \sin E - \sin(e \sin E)/[\sin \alpha - \sin E(1 - e \cos \alpha)]. \tag{7}$$

Letting α be a function of E according to Kepler's equation, we obtain, as a matter of course, exactly the same answer for the e^4-approximation as that above:

$$g' = g \text{ (G}' = \text{G)}, \qquad \text{Eq. (4)}. \tag{8}$$

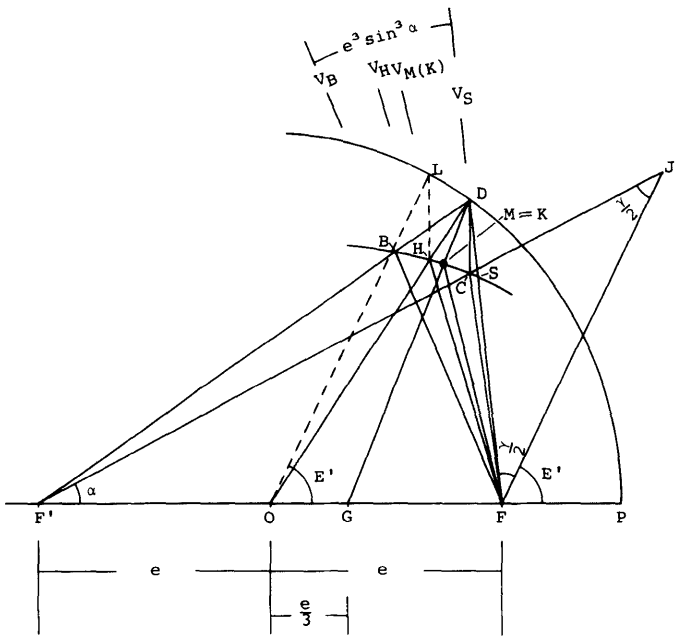

FIG. 2. P, perihelion; PSCMHB, elliptic planetary orbit; PDL, auxiliary circle; α, the given mean anomaly; OP = 1, F'J = 2; $E' = f(\alpha) = \angle$LOF, the angle introduced by Horrocks (ca. 1640) as an approximate substitute for the eccentric anomaly E in Kepler's equation (Fig. 1); D, Boulliau's (1657) auxiliary point; B,H,S, points of intersection of the orbit with the lines, DF', DO, DF, representing the planetary position, respectively, according to Boulliau (1657), Horrocks (ca. 1640), and our comparative computation. The true planetary position K lies precisely at M ($\simeq$K) between H and S (cf. Fig. 3).

(3) A Geometrical Consideration

Around 1640 Horrocks found an approximate solution, E', to the eccentric anomaly E of Kepler's equation (Fig. 2), certainly the first solution after Kepler. His substitute E' is given by a sum of the mean anomaly and one-half of the equation of the center γ as [5]

$$E' = \alpha + \gamma/2 = \alpha + \tan^{-1}[e \sin \alpha/(1 - e \cos \alpha)].$$

Likewise

$$\tan E' = \sin \alpha/(\cos \alpha - e); \tag{9}$$

hence,

$$E' = f(\alpha).$$

Since the angle DOF is given by

$$\tan \angle\text{DOF} = \rho \sin \alpha/[\sqrt{1 - e^2}\,(\rho \cos \alpha - e)] = \sqrt{1 - e^2} \sin \alpha/(\cos \alpha - e), \tag{10}$$

its eccentric anomaly LOF is simply

$$\angle \mathrm{LOF} = E'. \tag{11}$$

Hence the planetary position according to Horrocks is geometrically equivalent to the point H, the intersection of DO with the elliptic orbit. Horrocks' method then gives the true anomaly:

$$\tan(V_{\mathrm{H}}/2) = \sqrt{(1 + e)/(1 - e)}\,\tan(E'/2). \tag{12}$$

From (9) and (12) we obtain

$$V_{\mathrm{H}} = \alpha + (2e - e^3/8)\sin\alpha + (5/4)e^2 \sin 2\alpha + (25/24)e^3 \sin 3\alpha + \cdots; \tag{13}$$

hence, Horrocks' error against Kepler's true anomaly [6] is given by

$$V_{\mathrm{H}} - V_{\mathrm{K}} = \tfrac{1}{6}\, e^3 \sin^3\alpha. \tag{14}$$

According to Boulliau's hypothesis (1657) the planetary position is given by B, the intersection of DF′ with the elliptic orbit. By a similar procedure his error is then given by

$$V_{\mathrm{B}} - V_{\mathrm{K}} = \tfrac{2}{3}\, e^3 \sin^3\alpha. \tag{15}$$

Likewise, the error of the planetary position S, the intersection of DF with the orbit, is given by

$$V_{\mathrm{S}} - V_{\mathrm{K}} = -\tfrac{1}{3}\, e^3 \sin^3\alpha. \tag{16}$$

The above deviations of different geometrical points from Kepler's true planetary position are compared in Fig. 3.

From the above illustration it can easily be surmised that the true position to the e^3-approximation according to Kepler's equation must be situated on the line DG, where G is on the line OF at distance $e/3$ from the center. This surmise agrees with our accurate computations shown in (1) and (2) above.

(4) Geometrical Method 3

Putting g to the e^2-approximation according to (4) into (3), we obtain the error to the e^5-approximation as

$$\delta_g - \delta_{\mathrm{K}} \simeq \frac{e^5}{270}\sin^3 E(20 - 11\sin^2 E),$$

where we may replace $\sin E$ by $\sin\alpha$; hence,

$$(\delta_g - \delta_{\mathrm{K}})_{\mathrm{max,min}} \simeq \pm\frac{e^5}{30} \simeq \pm 0.05'' \text{ (Mars)},$$

$$\text{at } E \simeq \alpha \simeq \pm\pi/2 \text{ [}g\text{ of the }e\text{-order; (4)].} \tag{17}$$

Since the difference between $\cos E$ and $\cos\delta_g$ is bounded by the e-order

$$\cos E - \cos\delta_g = \frac{e}{3}\sin^2\alpha + \cdots, \tag{18}$$

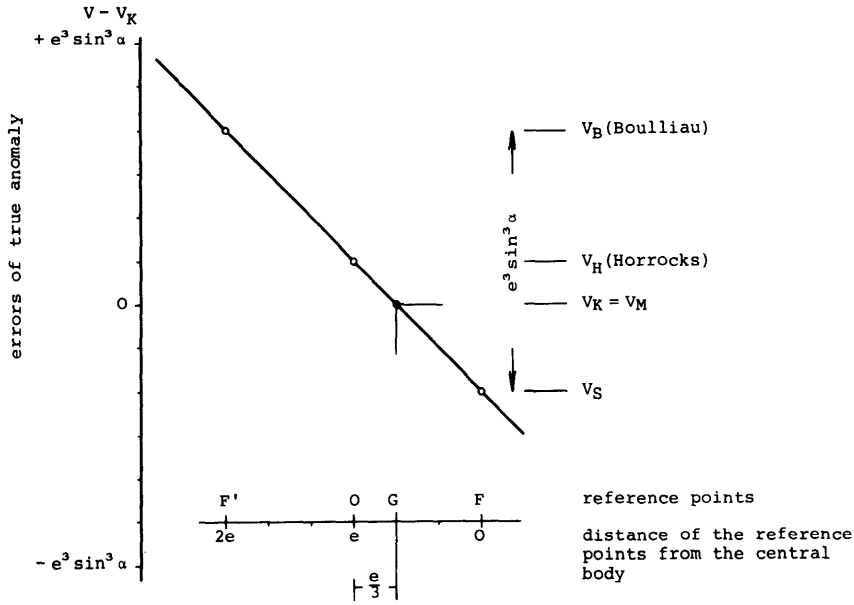

FIG. 3. In the range between two foci, F and F′, the error of the computing method based on the point D (Fig. 1) changes proportionally to the movement of the reference point on the apsidal line. In a computation to the e^3-approximation the error vanishes when the reference point is on the line OF at $e/3$ from the center O, agreeing with an accurate computation to the e^4-approximation [cf. (1) and (2)].

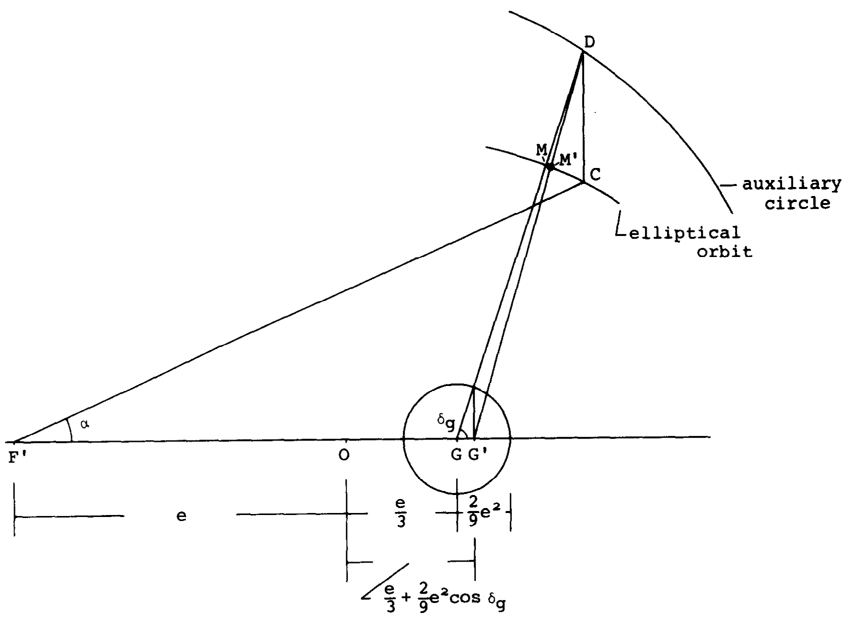

FIG. 4. The distance g [OG and OG′, (4), (19)] from the center satisfies the condition $\delta_g = \delta_K$ [(3)] to the e^3- and e^4-approximations. Thus the resulting geometrical points, M and M′, in the orbit represent the true planetary position to within ±0.6″ and ±0.05″ [Mars, (5), (17)] as a function of the mean anomaly $\alpha = \omega t$.

we may replace cos E in (4) for our order of approximation by cos δ_g; thus,

$$g \simeq \frac{e}{3} + \frac{2}{9} e^2 \cos \delta_g . \tag{19}$$

This approximation for g is now geometrically determinable by putting an epicycle with a radius of $(\frac{2}{9})e^2$ at $G_{e/3}$ (Fig. 4). Owing to (18) above, the error in the resulting geometrical position of the planet on the orbit M′ as seen from any point between the two foci F and F′ is bounded by the e^5-order quantity given by (17) ($\simeq 0.05''$ for Mars).

NOTES

1. For the first application, cf. Maeyama [1988b, Fig. 5.1].
2. For the solar parallax mentioned in the summary above, cf. Wilson [1969] and Maeyama [1974, 1975, 1988b].
3. For the geometrical application of the mean anomaly, cf. Maeyama [1988a].
4. For the following, cf. also Maeyama [1971, 35–36].
5. For this question, cf. Horrocks [1673] and Gaythorpe [1925].
6. For this problem, cf., e.g., Brouwer and Clemence [1961, 77], also Gaythorpe [1925].

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